

ON A PROBLEM À LA KUMMER-VANDIVER FOR FUNCTION FIELDS

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ABSTRACT. We use Artin-Schreier base change to construct counterexamples to a Kummer-Vandiver type question for function fields.

1. INTRODUCTION

Let p be a prime number and let F be the maximal real subfield of $\mathbb{Q}(\mu_p)$. The famous Kummer-Vandiver conjecture asserts that

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{Pic } O_F = \{0\}.$$

It has been verified for all p less than 163,577,856 [3]. However, heuristic arguments of Washington suggest that the number of counterexamples p up to X should grow as $\log \log X$, making it difficult to find either counterexamples or convincing numerical evidence towards the conjecture.

The second author has recently proven a function field analogue of the Herbrand-Ribet theorem, and formulated a version of the Kummer-Vandiver conjecture in this context [6]. In this note, which complements [6], we construct counterexamples to this Kummer-Vandiver statement.

Let us now recall the statement of this analogue. Let $A = \mathbb{F}_q[T]$ and let C be the Carlitz module over A . This is the A -module scheme over $\text{Spec } A$ whose underlying group scheme is the additive group $\mathbb{G}_{a,A}$, and on which A acts via the \mathbb{F}_q -algebra homomorphism

$$\phi: A \rightarrow \text{End } \mathbb{G}_{a,A}: T \mapsto T + \tau$$

where $\tau: \mathbb{G}_{a,A} \rightarrow \mathbb{G}_{a,A}$ is the q -th power Frobenius endomorphism. It is an example of a Drinfeld module, and in many ways it is an analogue of the multiplicative group in characteristic zero [4].

Let $P \in A$ be monic irreducible, let $k = \mathbb{F}_q(T)$ be the fraction field of A and let K/k be the extension obtained by adjoining the P -torsion points of C . Then K/k is Galois and there is a canonical isomorphism

$$\omega_P: \text{Gal}(K/k) \rightarrow (A/P)^\times,$$

which one can think of as the mod P Teichmüller character. Let R be the integral closure of A in K and put $Y = Y_P = \operatorname{Spec} R$.

Let $C[P]$ be the P -torsion subscheme of C and let $C[P]^D$ be its Cartier dual. Consider the flat cohomology group

$$H_P := H^1(Y_{P,\mathbb{A}}, C[P]^D).$$

This is an A/P -vector space on which the Galois group $\operatorname{Gal}(K/k)$ acts, so it decomposes in isotypical components as

$$H_P = \bigoplus_{n=1}^{q^{\deg P} - 1} H_P(\omega_P^n).$$

In [6], it is shown that for n in the range $1 \leq n < q^{\deg P} - 1$ which are divisible by $q - 1$ one has

$$H_P(\omega_P^{n-1}) \neq \{0\} \quad \text{if and only if} \quad B(n) \equiv 0 \pmod{P},$$

where $B(n) \in k$ is the n -th Bernoulli-Carlitz number. The Kummer-Vandiver problem can be stated as follows (see [6, Question 1]):

Question 1. *Is $H_P(\omega_P^{n-1}) = \{0\}$ for n not divisible by $q - 1$?*

The analogy with the classical Kummer-Vandiver conjecture is (implicitly) explained in [6, Remark 2]: using the Kummer sequence and flat duality it is shown that the classical Kummer-Vandiver conjecture is equivalent with the statement that

$$H^1((\operatorname{Spec} \mathbb{Z}[\zeta_\ell])_{\mathbb{A}}, \mu_\ell^D)(\chi_\ell^{n-1}) \stackrel{?}{=} 0 \text{ if } n \text{ is odd,}$$

where ℓ is an odd prime, ζ_ℓ a primitive ℓ -th root of unity and χ_ℓ denotes the mod ℓ cyclotomic character.

In this paper we use Artin-Schreier change of variables and computer calculations to construct counterexamples to the above statement. For example, we use properties of the prime

$$P = T^3 - T^2 + 1 \in \mathbb{F}_3[T]$$

to show that the prime

$$Q = P(T^3 - T) = T^9 - T^6 - T^4 - T^3 - T^2 + 1$$

satisfies $H_Q(\omega_Q^{9840}) \neq 0$. Note that $9840 = n - 1$ with $n = (q^{\deg Q} - 1)/2$.

The degree of the prime Q is too high to allow for a direct computation of H_Q .

In a forthcoming paper we compare the flat cohomology groups of [6] with the group of “units modulo circular units” introduced by Anderson [1], and show amongst other things that the Kummer-Vandiver problem of [6] is equivalent with Anderson’s Kummer-Vandiver conjecture [1,

§4.12]. In particular, the present counterexamples will also serve as counterexamples to Anderson's conjecture.

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2. NOTATION

2.1. L -functions. Let F/E be a finite abelian extension of function fields of curves over \mathbb{F}_q . Assume that \mathbb{F}_q is algebraically closed in both E and F . Let $\chi: \text{Gal}(F/E) \rightarrow \mathbb{C}^\times$ be a homomorphism, and let $E_\chi \subset F$ be the fixed field of $\ker \chi$. We set

$$L(X, E, \chi) = \prod_{v \text{ place of } E} (1 - \chi(v)X^{\deg v})^{-1} \in \mathbb{C}[[X]],$$

where $\chi(v) = \chi((v, E_\chi/E))$ if v is unramified in E_χ/E and $\chi(v) = 0$ otherwise. Here $(-, E_\chi/E)$ denotes the global reciprocity map. Recall that $L(X, E, \chi)$ is a rational function and that if $\chi \neq 1$ then $L(X, E, \chi)$ is a polynomial whose coefficients are algebraic integers.

2.2. The cyclotomic function fields. Let p be a prime number. Let \mathbb{F}_q be a finite field having q elements, $q = p^s$, where p is the characteristic of \mathbb{F}_q . Let $A = \mathbb{F}_q[T]$ be the polynomial ring in one variable T and let $k = \mathbb{F}_q(T)$ be its field of fractions. We denote the set of monic elements in A by A_+ . For $n \geq 0$, we denote the set of elements in A_+ of degree n by A_n . We fix \bar{k} , an algebraic closure of k . All finite extensions of k considered in this note are assumed to be contained in \bar{k} . We denote the unique place of k which is a pole of T by ∞ .

Let $P \in A$ be monic irreducible of degree d . We denote the P -th cyclotomic function field by K_P (see [4], chapter 7). Recall that K_P/k is the maximal abelian extension of k such that:

- (1) K_P/k is unramified outside P and ∞ ,
- (2) K_P/k is tamely ramified at P and ∞ ,
- (3) for every place v of K_P above ∞ , the completion of K_P at v is isomorphic to $\mathbb{F}_q((\frac{1}{T}))^{(q-1)\sqrt{-T}}$.

The Galois group $\text{Gal}(K_P/k)$ is canonically isomorphic with $(A/PA)^\times$ and the subgroup $\mathbb{F}_q^\times \subset (A/PA)^\times$ is both the inertia and the decomposition group of ∞ in K_P/k .

3. CYCLICITY OF DIVISOR CLASS GROUPS

3.1. An Artin-Schreier extension and the function γ . Let $i: A_+ \rightarrow \mathbb{Z}/p\mathbb{Z}$ be the function that maps a polynomial

$$T^n + \alpha_1 T^{n-1} + \cdots + \alpha_n$$

to $\text{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \alpha_1 \in \mathbb{Z}/p\mathbb{Z}$. Observe that for all $a, b \in A_+$ we have $i(ab) = i(a) + i(b)$.

Let $\theta \in \tilde{k}$ be a root of $X^p - X = T$. Then the extension \tilde{k} obtained by adjoining θ to k is rational and we have $\tilde{k} = \mathbb{F}_q(\theta)$. The extension ramifies only at ∞ . The integral closure of A in \tilde{k} , which we denote by \tilde{A} , is the polynomial ring $\mathbb{F}_q[\theta]$ in θ .

We have an isomorphism of groups $\mathbb{Z}/p\mathbb{Z} \rightarrow \text{Gal}(\tilde{k}/k)$ given by

$$n \mapsto \sigma_n = [\theta \mapsto \theta - n].$$

Let $(-, \tilde{k}/k)$ be the Artin symbol for ideals, then for all $a \in A_+$ we have [2, Lemma 2.1]

$$(1) \quad (aA, \tilde{k}/k) = \sigma_{i(a)}.$$

Let n be a positive integer. By [2, Lemma 3.2] for all m sufficiently large we have

$$\sum_{a \in A_m} i(a) a^n = 0.$$

We define

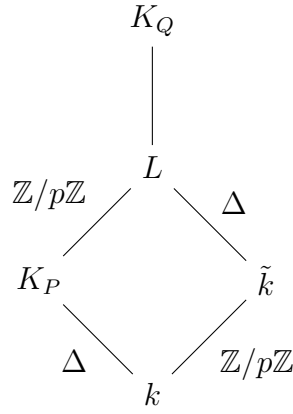
$$\gamma(n) := \sum_{m \geq 0} \sum_{a \in A_m} i(a) a^n \in A.$$

3.2. Cyclotomic extensions of k and of \tilde{k} . Now, we fix a prime P in A of degree d such that $i(P) \neq 0$. Set $Q(\theta) := P(T) \in \tilde{A}$. Note that by (1) the polynomial $Q(\theta) \in \mathbb{F}_q[\theta]$ is irreducible. Its degree is pd .

Let K_P be the P -th cyclotomic function field for A , with Galois group $\Delta = (A/PA)^\times$, and let \tilde{K}_Q be the Q -th cyclotomic function field for \tilde{A} , with Galois group $\tilde{\Delta} = (\tilde{A}/Q\tilde{A})^\times$. By [2, Lemma 2.2] we have:

$$K_P \subset \tilde{K}_Q.$$

Let L be the compositum of \tilde{k} and K_P inside \tilde{K}_Q . Then L is an abelian extension of k with Galois group $\mathbb{Z}/p\mathbb{Z} \times \Delta$.



The inclusion $A/PA \subset \tilde{A}/Q\tilde{A}$ induces an injective homomorphism $\Delta \rightarrow \tilde{\Delta}$. On the other hand, we can identify the Galois group of L over \tilde{k} with Δ , and obtain a surjective map

$$\tilde{\Delta} \rightarrow \Delta,$$

which is explicitly given by

$$(\tilde{A}/Q\tilde{A})^\times \rightarrow (A/PA)^\times : a \mapsto \text{Nm}_{\tilde{k}/k} a.$$

3.3. Comparison of L -functions. Let W_0 be the ring of Witt vectors of A/QA and let $W = W_0[\zeta_p]$, where ζ_p is a primitive p -th root of unity. Let $\omega_P : \Delta \rightarrow W^\times$ and $\omega_Q : \tilde{\Delta} \rightarrow W^\times$ be the Teichmüller characters. We will denote by $\widetilde{\omega}_P$ the same character as ω_P , but seen as a character on $\text{Gal}(L/\tilde{k})$. In particular, we have

$$\widetilde{\omega}_P = \omega_Q^{\frac{q^{pd}-1}{q^d-1}}.$$

Let n be an integer such that $1 \leq n \leq q^d - 2$. Then [2, Lemma 2.4]

$$(2) \quad L(X, L/\tilde{k}, \widetilde{\omega}_P^n) = \prod_{\phi} L(X, L/k, \phi \omega_P^n),$$

where ϕ runs over all characters of $\text{Gal}(\tilde{k}/k) = \mathbb{Z}/p\mathbb{Z}$.

Observe that, if $\phi \neq 1$, then $L(X, \phi \omega_P^n)$ is a polynomial of degree d (apply [2], Lemma 2.3 for both A and \tilde{A}). Furthermore, we have:

$$(3) \quad L(X, L/k, \psi \omega_P^n) = \sum_{m=0}^d \left(\sum_{a \in A_m} \zeta_p^{i(a)} \omega_P(a)^n \right) X^m,$$

where $\psi : \mathbb{Z}/p\mathbb{Z} \rightarrow W^\times$ is the character that maps 1 to ζ_p .

3.4. Congruences. Assume that n is not divisible by $q - 1$. Then the Bernoulli-Goss polynomial $\beta(n)$ is defined as follows:

$$\beta(n) = \sum_{m \geq 0} \sum_{a \in A_m} a^n \in A.$$

(The inner sum vanishes for all sufficiently large m .)

Proposition 1. *Assume that n is not divisible by $q - 1$ and that p is odd. Then the following are equivalent:*

- (1) $v_p(L(1, L/k, \psi\omega_P^n)) \geq 2/(p - 1)$,
- (2) $\beta(n)$ and $\gamma(n)$ are divisible by P .

Proof. Using the congruence

$$\zeta_p^i \equiv 1 + i(\zeta_p - 1) \pmod{(\zeta_p - 1)^2}$$

we deduce from (3) the congruence

$$L(1, L/k, \psi\omega_P^n) \equiv L(1, K_P/k, \omega_P^n) + (\zeta_p - 1) \left(\sum_{m=0}^d \sum_{a \in A_m} i(a) \omega_P(a)^n \right)$$

modulo $(\zeta_p - 1)^2$. Since $L(1, K_P/k, \omega_P^n) \in W_0$ and p is odd, it follows that $L(1, L/k, \psi\omega_P^n)$ vanishes modulo $(\zeta_p - 1)^2$ if and only if both

$$L(1, K_P/k, \omega_P^n) \equiv 0 \pmod{p}$$

and

$$\sum_{m=0}^d \sum_{a \in A_m} i(a) \omega_P(a)^n \equiv 0 \pmod{\zeta_p - 1}.$$

The first congruence holds if and only if P divides $\beta(n)$ and the second if and only if P divides $\gamma(n)$. \square

3.5. Divisor class groups. Let E be a finite extension of k , with constant field \mathbb{F}_{q^n} . We have an exact sequence

$$0 \rightarrow \mathbb{F}_{q^n}^\times \rightarrow E^\times \rightarrow \text{Div}^0 E \rightarrow \text{Cl } E \rightarrow 0$$

where $\text{Div}^0 E$ is the group of degree 0 divisors on E and $\text{Cl } E$ the group of divisor classes of degree 0 of E . Since W is flat over \mathbb{Z} this sequence remains exact after tensoring with W , and since $\mathbb{F}_{q^n}^\times$ has no p -torsion we obtain a short exact sequence

$$(4) \quad 0 \rightarrow W \otimes E^\times \rightarrow W \otimes \text{Div}^0 E \rightarrow C(E) \rightarrow 0,$$

where $C(E) = W \otimes \text{Cl } E$.

Proposition 2. *Let F/E be a finite Galois extension with Galois group G . Then there is a natural short exact sequence*

$$0 \rightarrow C(E) \rightarrow C(F)^G \rightarrow W \otimes \frac{(\operatorname{Div} F)^G}{\operatorname{Div} E}$$

and $(\operatorname{Div} F)^G / \operatorname{Div} E$ is generated by the ramified primes.

Proof. By Hilbert 90 we have $H^1(G, F^\times) = 0$, and since W is flat over \mathbb{Z} we have

$$H^1(G, W \otimes F^\times) = W \otimes H^1(G, F^\times) = 0.$$

Taking G -invariants in the sequence (4) for F gives a short exact sequence

$$0 \rightarrow W \otimes E^\times \rightarrow W \otimes (\operatorname{Div}^0 F)^G \rightarrow C(F)^G \rightarrow 0.$$

Comparing this with (4) for E gives the desired exact sequence. \square

Corollary 1. *Assume that n is not divisible by $q - 1$. Then*

$$C(K_P)(\omega_P^{-n}) = C(L)(\widetilde{\omega}_P^{-n})^{\operatorname{Gal}(L/K_P)}$$

and

$$C(L)(\widetilde{\omega}_P^{-n}) = C(\widetilde{K}_Q)(\omega_Q^{-n(q^{pd}-1)/(q^d-1)}).$$

Proof. Since L/K_P is unramified away from the primes above ∞ , we have that $(\operatorname{Div} L)^G / \operatorname{Div} K_P$ is generated by the primes above ∞ . Let S be the set of primes of L above ∞ and W^S the free W -module with basis S . Because n is not divisible by $q - 1$ we have

$$W^S(\widetilde{\omega}_P^{-n}) = 0,$$

hence the first claim follows from the Proposition. For the second claim, use that \widetilde{K}_Q/L is unramified away from Q and the primes above ∞ , and that Q is totally ramified. \square

Theorem 1. *Assume $p \neq 2$. Let $P \in A$ be monic irreducible of degree d , and such that $i(P) \neq 0$. Let n be an integer such that $1 \leq n \leq q^d - 2$, not divisible by $q - 1$ and such that $\beta(n)$ and $\gamma(n)$ are divisible by P . Then $C(\widetilde{K}_Q)(\omega_Q^{-n(q^{pd}-1)/(q^d-1)})$ is not cyclic.*

Proof. Set

$$U = C(L)(\widetilde{\omega}_P^{-n}) = C(\widetilde{K}_Q)(\omega_Q^{-n(q^{pd}-1)/(q^d-1)})$$

and assume that U is W -cyclic, but that $\beta(n)$ and $\gamma(n)$ are divisible by P .

Since $\beta(n)$ is divisible by P , it follows that $U^{\operatorname{Gal}(L/K_P)} = C(K_P)(\omega_P^{-n})$ is nonzero, and in particular that U is nonzero. Let $x \in U$ be a generator, so that $U = Wx$. Let g be a generator of $\operatorname{Gal}(L/K_P)$.

We have $gx = wx$ for some $w \in W^\times$. This implies that $w^p x = x$, and it follows that $w^p - 1 \equiv 0 \pmod{p}$ and $w \equiv 1 \pmod{p}$. Since $v_p(1 + w + \cdots + w^{p-1}) = 1$ we find

$$pU = (1 + w + \cdots + w^{p-1})U \subset U^{\text{Gal}(L/K_P)}$$

and therefore the length of $U/U^{\text{Gal}(L/K_P)}$ is at most 1.

On the other hand, by [5] and Corollary 1, we have that the length of $U/U^{\text{Gal}(L/K_P)}$ equals

$$v_p(L(1, L/\tilde{k}, \widetilde{\omega_P^n})) - v_p(L(1, K_P/k, \omega_P^n))$$

and by (2) this equals

$$(p-1)v_p(L(1, L/k, \psi_P^n)).$$

From Proposition 1 we deduce that the length of $U/U^{\text{Gal}(L/K_P)}$ is at least 2, a contradiction. \square

4. KUMMER-VANDIVER

If $P \in A$ is monic irreducible we write Y_P for the spectrum of the integral closure of A in K_P .

Theorem 2. *Assume $p \neq 2$. Let $P \in A$ be monic irreducible of degree d and such that $i(P) \neq 0$. Let n be an integer such that*

- (1) $\beta(n)$ is divisible by P if n is not divisible by $q-1$;
- (2) $\gamma(n)$ is divisible by P .

Let $Q(T) = P(T^p - T)$ and $N = n(q^{pd} - 1)/(q^d - 1)$. Then Q is irreducible in A and

$$H^1(Y_{Q,\text{fl}}, C[Q]^D)(\omega_Q^{-N-1}) \neq \{0\}.$$

Proof. We split the proof in cases depending on the divisibility of n and dn by $q-1$. Note that N is divisible by $q-1$ if and only if nd is divisible by $q-1$.

Case 1. Assume that n is divisible by $q-1$. This case is treated in [2]. By [2, Proposition 2.6], we get

$$(W \otimes \text{Pic } Y_Q)(\omega_Q^{-N}) \neq \{0\}.$$

Without loss of generality we may assume that $1 \leq n < q^d - 1$. Then by the work of Okada ([7], see also [4, §8.20]):

$$B(q^{pd} - 1 - N) \equiv 0 \pmod{Q(T)}.$$

By Theorem 1 of [6] (the ‘‘Herbrand-Ribet theorem’’) we conclude

$$H^1(Y_{Q,\text{fl}}, C[Q]^D)(\omega_Q^{-N-1}) \neq \{0\}.$$

Case 2. Now assume that n is not divisible by $q - 1$ but dn is. Then by Theorem 1 the module $C(K_Q)(\omega_Q^{-N})$ is not cyclic, and so we must have:

$$(W \otimes \text{Pic } Y_Q)(\omega_Q^{-N}) \neq \{0\}.$$

We conclude with the same argument as in case 1.

Case 3. Assume that nd is not divisible by $q - 1$. As in the previous case, we find that

$$(W \otimes \text{Pic } Y_Q)(\omega_Q^{-N}) \neq \{0\}.$$

Now we conclude with a different argument. By the above non vanishing, and by exact sequence (2) of [6] we find that the space of Cartier-invariant ω^{-N} -typical differential forms

$$A/Q \otimes_{\mathbb{F}_q} \Gamma(Y, \Omega_Y)^{c=1}(\omega^{-N})$$

is at least two-dimensional. With the exact sequence of Theorem 2 of *loc. cit.* one concludes that

$$H^1(Y_{Q,\mathbb{A}}, C[Q]^D)(\omega^{-N-1}) \neq \{0\}.$$

(Using the fact that the ω^{-N} -part of the last module of the exact sequence of Theorem 2 is 1-dimensional. Note that the same argument is used in the proof of Theorem 1 of *loc. cit.*, see [6, §4].) \square

4.1. An example. Let $q = 3$. One can verify that with $P = T^3 - T^2 + 1$ and $n = 13$ the conditions of Theorem 2 are satisfied. Indeed, one has

$$\beta(13) = -T^9 - T^3 - T + 1$$

and

$$\gamma(13) = -T^{12} - T^{10} + T^9 - T^4 + T^3 + T - 1,$$

both of which are divisible by $P = T^3 - T^2 + 1$ in $\mathbb{F}_3[T]$. Also, note that n is not divisible by $q - 1$. Using Theorem 2 we thus find that the prime

$$Q(T) = P(T^3 - T) = T^9 - T^6 - T^4 - T^3 - T^2 + 1$$

satisfies

$$H^1(Y_{Q,\mathbb{A}}, C[Q]^D)(\omega^{-N-1}) \neq \{0\}$$

where we have

$$N = n \frac{q^{pd} - 1}{q^d - 1} = 9841.$$

This is the counterexample to the analogue of the Kummer-Vandiver conjecture stated at the end of the introduction (we have $-9842 \equiv 9840 \pmod{q^{pd} - 1}$.)

5. CHARACTERISTIC $p = 2$

We now assume that $p = 2$. With some minor changes, the above arguments still work, but the result is weaker.

We keep the notations of section 3. If $P(T)$ is a prime in A of degree d such that $i(P) \neq 0$, then $Q(\theta) = P(T)$ is a prime of degree $2d$ in $\tilde{A} = \mathbb{F}_q[\theta]$, where $\theta^2 - \theta = T$. Set again $L = \tilde{k}K_P \subset \tilde{K}_Q$. We have the following version of Theorem 1.

Theorem 3. *Assume $p = 2$. Let $P \in A$ be monic irreducible of degree d , and such that $i(P) \neq 0$. Let n be an integer such that $1 \leq n \leq q^d - 2$, not divisible by $q - 1$ and such that $\gamma(n)$ is divisible by P and $L(1, K_P/k, \omega_P)$ is divisible by 4. Then $C(K_Q)(\omega_Q^{-n(q^{pd}-1)/(q^d-1)})$ is not cyclic.*

Note that $\beta(n)$ is divisible by P if and only if $L(1, K_P/k, \omega_P)$ is divisible by 2, so the hypothesis are stronger than those in Theorem 1.

Proof of Theorem 3. The proof is almost identical to that of Theorem 1. Let ψ be the unique non-trivial character of $G = \text{Gal}(\tilde{k}/k)$.

Proposition 1 does no longer hold, since we no longer have that $(\zeta_p - 1)^2$ divides p . However, if $\gamma(n)$ is divisible by P and if $L(1, K_P/k, \omega_P)$ is divisible by 4 (instead of 2), we can still conclude

$$v_p(L(1, L/k, \psi\omega_P^n)) \geq 2.$$

Denote the length of $U = C(L)(\tilde{\omega}_P^{-n})$ by N . We have

$$N = v_p(L(1, L/k, \psi\omega_P^n)) + v_p(L(1, L/K, \omega_P^n)) \geq 4.$$

Let g be the nontrivial element of $\text{Gal}(L/K_P)$. Then the length of U^g equals $v_p(L(1, L/K, \omega_P^n))$, which by hypothesis is at least 2.

Suppose that U is a cyclic W -module, and let $x \in U$ be a generator, so that $U = Wx$. There is a $w \in W^\times$ so that $gx = wx$. We then have that $w^2 - 1$ is divisible by 2^N . We find that $w - 1$ is divisible by 2^{N-1} but not by 2^N (since $U^g \neq U$.) It follows that $(1 + g)U = 2U$, and as in the proof of Theorem 1 we conclude that U/U^g has length at most 1, contradicting our hypothesis. We conclude that U cannot be W -cyclic. \square

Using this, we get the following variation of Theorem 2 in characteristic 2, with the same proof.

Theorem 4. *Assume $p = 2$. Let $P \in A$ be monic irreducible of degree d and such that $i(P) \neq 0$. Let n be an integer such*

- (1) $L(1, \omega_P^n)$ is divisible by 4 if n is not divisible by $q - 1$;
- (2) $\gamma(n)$ is divisible by P .

Let $Q(T) = P(T^2 - T)$ and $N = n(q^d + 1)$. Then Q is irreducible in A and

$$H^1(Y_{Q,\text{fl}}, C[Q]^D)(\omega^{-N-1}) \neq \{0\}.$$

5.1. An example. Let $q = 4$ and $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$. Then $P = T^5 + \alpha^2 T^4 + T^3 + \alpha T^2 + \alpha^2$ with $n = 341 = (4^5 - 1)/(4 - 1)$ satisfies the hypothesis, leading to a counterexample $Q \in \mathbb{F}_4[T]$ to Question 1.

6. HEURISTICS

This section contains no mathematical theorems, but only crude heuristic arguments and numerical observations. Our main goal is to convince the reader that one could *a priori* expect to construct many counterexamples using the above base change strategy.

The arguments are specific to *odd* q so we assume throughout the section that q is odd.

We argue that one could expect that Theorem 2 yields at least $cX^{1/p}(\log X)^{-1}$ counter-examples of residue cardinality at most X to Question 1 (for some constant $c > 0$), which is much more than the $\log \log X$ counter-examples predicted by Washington's heuristics.

In fact we will only consider counter-examples of a particular form. Assume that q is odd. Note that by Theorem 2, if we are given

- (1) an integer m with $1 \leq m < q - 1$;
- (2) a monic irreducible $P \in A$ of degree d ,

such that

- (1) $q - 1$ does not divide md ;
- (2) $\beta(m(q^d - 1)/(q - 1)) \equiv \gamma(m(q^d - 1)/(q - 1)) \equiv 0 \pmod{P}$;

then $Q(T) = P(T^p - T)$ is monic irreducible of degree pd and

$$H^1(Y_{Q,\text{fl}}, C[Q]^D)(\omega^{1+m(q^{pd}-1)/(q-1)}) \neq \{0\},$$

giving a counterexample to Question 1.

The reason to restrict to n of the form $m(q^d - 1)/(q - 1)$ lies in the following trivial observation:

Lemma 1. *Let $P \in A$ be irreducible of degree d . If n is a multiple of $(q^d - 1)/(q - 1)$ then $\beta(n)$ and $\gamma(n)$ modulo P lie inside $\mathbb{F}_q \subset A/P$. \square*

So we may expect that $\beta(n)$ and $\gamma(n)$ are much more likely to vanish modulo a prime P of degree d if n is of the form $m(q^d - 1)/(q - 1)$.

Assume that $q - 1$ does not divide md . We make the following hypotheses on a “random” monic irreducible P of degree d :

- (1) $i(P)$ is non-zero with probability $(p - 1)/p$;
- (2) $\beta(m(q^d - 1)/(q - 1))$ is zero modulo P with probability $1/q$;

- (3) $\gamma(m(q^d - 1)/(q - 1))$ is zero modulo P with probability $1/q$;
- (4) the above probabilities are independent of each other, and independent of the vanishing of $i(P)$.

The first hypothesis is essentially an instance of the Chebotarev density theorem, the second and the third are motivated by Lemma 1, and the fourth is nothing more than wishful thinking. To some extent one can verify these statements experimentally. In Table 1 we reproduce some numerical data regarding these hypotheses. Note that in the example of Table 1 β seems to have a slight bias towards vanishing, we have no explanation for this bias.

Finally, we show that under the above hypothesis, for some $c > 0$ we find that for all X sufficiently large there are at least $cX^{1/p}(\log X)^{-1}$ primes of residue cardinality at most X that contradict Question 1.

Indeed, for all X sufficiently large there is a positive integer d with

$$(\log_q X^{1/p}) - 2 < d \leq \log_q X^{1/p}$$

and d not divisible by $q - 1$. Taking $m = 1$, we should find

$$\frac{p-1}{p} \cdot \frac{1}{q} \cdot \frac{1}{q} \cdot \frac{q^d}{d} \geq c \frac{X^{1/p}}{\log X}$$

monic irreducibles P of degree d satisfying the conditions (with $c > 0$ independent of X). Each of these leads to a counter-example Q of residue cardinality at most X .

d	$\beta(n) \equiv 0 \ (P)$	$\gamma(n) \equiv 0 \ (P)$	$\beta(n) \equiv \gamma(n) \equiv 0 \ (P)$
9	428	318	142
11	395	344	137
13	416	332	147

TABLE 1. The number of P satisfying various congruences, out of random samples of 1000 primes P in $\mathbb{F}_3[t]$ with $i(P) \neq 0$, of degrees 9, 11 and 13. Again $n = (q^d - 1)/2$. Note that every P counted in the rightmost column gives rise to a prime Q which gives a counterexample to Question 1.

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